

LONG-TERM STABILITY ESTIMATES AND EXISTENCE OF A GLOBAL ATTRACTOR IN A FINITE ELEMENT APPROXIMATION OF THE NAVIER-STOKES EQUATIONS WITH NUMERICAL SUB-GRID SCALE MODELING

SANTIAGO BADIA*, RAMON CODINA†, AND JUAN VICENTE GUTIÉRREZ-SANTACREU ‡

Abstract. Variational multiscale methods lead to stable finite element approximations of the Navier-Stokes equations, both dealing with the indefinite nature of the system (pressure stability) and the velocity stability loss for high Reynolds numbers. These methods enrich the Galerkin formulation with a sub-grid component that is modeled. In fact, the effect of the sub-grid scale on the captured scales has been proved to dissipate the proper amount of energy needed to approximate the correct energy spectrum. Thus, they also act as effective large-eddy simulation turbulence models and allow to compute flows without the need to capture all the scales in the system. In this article, we consider a dynamic sub-grid model that enforces the sub-grid component to be orthogonal to the finite element space in L^2 sense. We analyze the long-term behavior of the algorithm, proving the existence of appropriate absorbing sets and a compact global attractor. The improvements with respect to a finite element Galerkin approximation are the long-term estimates for the sub-grid component, that are translated to effective pressure and velocity stability. Thus, the stabilization introduced by the sub-grid model into the finite element problem is not deteriorated for infinite time intervals of computation.

Key words. Navier-Stokes problem, long-term stability, absorbing set, global attractor, stabilized finite element methods, sub-grid scales

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1. Introduction. The dynamics of Newtonian incompressible flows is governed by the Navier-Stokes equations, a dynamical system that consists in a set of nonlinear partial differential equations with a dissipative structure. For two-dimensional problems, the energy of this system has been proved to be bounded by the data (external forces and boundary conditions) for all times. It is also possible to bound the $H^1(\Omega)$ -norm of the fluid velocity, which, together with the Rellich-Kondrachov theorem, allows to prove that any fluid velocity orbit converges to a finite dimensional set, the so-called global attractor, as the time variable goes to infinity (see [19, 37]). Fractal and Hausdorff dimensions of the global attractor have been estimated using Lyapunov exponents in dimension 2 and 3 [15, 20].

An accurate numerical approximation of the Navier-Stokes equations should mimic their long-term behavior. For direct numerical simulation (DNS), a crude Galerkin approximation using inf-sup stable finite elements admits a numerical global attractor, whose dimension has been estimated in [33]. The convergence of the numerical global attractor to the one of the Navier-Stokes equations has been analyzed in [26]. Similar results have been proved for finite differences [38].

The finite element approximation of the Navier-Stokes equations for large Reynolds numbers (Re) presents two main difficulties that can make their numerical approximations meaningless: one is the indefinite nature of the system, and the other the stability loss due to convection dominant regimes. The first problem can be cured by using appropriate velocity-pressure finite element spaces satisfying a discrete version of the Ladyzhenskaya-Babuška-Brezzi condition (see [5]). These finite element pairs are usually called inf-sup stable ele-

*International Center for Numerical Methods in Engineering (CIMNE), Universitat Politècnica de Catalunya, Jordi Girona 1-3, Edifici C1, 08034 Barcelona, Spain. sbadia@cimne.upc.edu. Santiago Badia acknowledges the support of the European Community through the Marie Curie contract NanoSim (MOIF-CT-2006-039522).

†International Center for Numerical Methods in Engineering (CIMNE), Universitat Politècnica de Catalunya, Jordi Girona 1-3, Edifici C1, 08034 Barcelona, Spain. ramon.codina@upc.edu

‡ Dpto. de Matemática Aplicada I, Universidad Sevilla, E. T. S. I. Informática. Avda. Reina Mercedes, s/n. 41012 Sevilla, Spain, Spain. juanvi@us.es

ments, and do not include many spaces that would be interesting for their simplicity and/or efficiency. When using Galerkin approximations and finite elements, the only way to solve the velocity stability loss is to capture all the spatial scales of the flow, i.e. to reduce the computational mesh size up to the Kolmogorov microscale λ_K , below which there are the smallest dissipative structures of the flow. This approach, known as direct numerical simulation, requires in dimension 3 $\mathcal{O}(\text{Re}^{2.25})$ mesh nodes. Unsurprisingly, this dimension is also related to the dimension of the continuous global attractor (see [15, 20, 37]). The memory usage grows so fast with respect to Re that DNS computations are unaffordable in most industrial applications, even at moderate Reynolds numbers. Anyway, DNS is a valuable tool in theoretical turbulence research: it allows a deeper understanding of this phenomenon and helps to validate turbulence models.

Both pressure instability and velocity stability loss for convection dominant regimes can be solved by using finite element stabilization techniques (see e.g. [6, 29, 8, 10, 14, 2]). In fact, stabilization is essential for the finite element approximation of high Re flows. The common feature of this family of algorithms is to introduce consistent terms to the formulation that would improve the stability properties of the numerical system without spoiling accuracy. Initially, these stabilization techniques were developed without a sound motivation till they were justified by a multiscale decomposition of the continuous solution into resolved (finite element) and unresolved (sub-grid) scales. Using this decomposition in the variational form of the problem, and modeling the effect of the subscales into the finite element problem, we end up with numerical methods that exhibit enhanced stability properties. We refer to [28, 30] for a detailed exposition of this approach, coined the variational multiscale (VMS) method. Applied to the Navier-Stokes equations, stabilized finite elements lead to stable formulations without the need of representing all the scales of the flow. Thus, coarser meshes can be used, drastically reducing the computational effort of DNS.

VMS sub-grid scale models have been motivated by numerical purposes (stability and convergence of the numerical algorithms), but they have also been proved to introduce a numerical dissipation that approximates well the physical dissipation at the unresolved scales [22, 10, 14, 27, 16, 35, 3]. So, these methods can be understood as large-eddy simulation (LES) turbulence models that properly account for the effect of the smaller universal scales onto the large scale motions of the flow that can be captured by the mesh.

The VMS framework is clear for linear stationary problems, leading to effective and accurate numerical methods. In those methods, the sub-grid component is modeled using local problems (the global sub-grid problem is localized at every finite element of the mesh) and the differential operator that defines the problem is replaced by an algebraic one (motivated by Fourier analysis in our case). As a result, the sub-grid component is approximated at every finite element as a closed form in terms of the finite element residual. Enforcing the sub-grid component to be orthogonal to the finite element space we recover the orthogonal sub-grid scale (OSS) model proposed by Codina in [8, 10], otherwise we get the algebraic sub-grid scales (ASGS) model, in the terminology of [10]. OSS has been proved to introduce less numerical dissipation than ASGS in [8].

The extension of this framework to transient and nonlinear problems is not obvious. The main difficulties lie in how to approximate the sub-grid time derivative in the sub-grid problem and how to track the subscale in the nonlinear iterative process. A straightforward choice for the time discrete system is to treat the time derivative of the sub-grid component as a reaction-like term, with reaction coefficient δt^{-1} , δt being the time step size. When $\delta t \rightarrow 0$, the algorithm tends to the non-stabilized Galerkin formulation, with the problems pointed out above. In [2, 13, 14] we have devised two cures to this instabilities. The first solution is to use OSS formulations together with a quasi-static approximation of the sub-grid

scales, i.e. the sub-grid time derivative is neglected and the steady-state sub-grid model used. A more consistent approach is to consider dynamic sub-grid models that keep the sub-grid time derivative. In this case, the sub-grid model turns into an ordinary differential equation. Dynamic subscales have been proved to exhibit unconditional stability properties in strong norms for convection-diffusion and Stokes systems (both for ASGS and OSS formulations), the semi-discrete problem in space is well-posed and space and time integration commute. We refer to [2, 13, 14] for some works showing the benefits of using dynamic sub-grid scales and numerical analyses that prove the unconditional stability and convergence of the method for some linear problems. With regard to turbulence modeling, the VMS dynamic sub-grid model is able to represent backscatter [35].

Summarizing, finite element formulations with VMS sub-grid models have been proved to be effective and accurate techniques for the numerical approximation of turbulent flows. These methods are not plagued by pressure instability or velocity stability loss and properly account for the energy dissipation at the unresolved scales. In particular, dynamic sub-grid models have solved the inconsistencies and instabilities of previous approaches. In this work, we analyze the long-term behavior of the semi-discrete (discrete in space) system that arises from the finite element approximation of the Navier-Stokes equations with a VMS dynamic OSS model. We will show that this LES method exhibits good stability properties at infinite time values, improving those of the Galerkin approximation. As we will show, the stabilization mechanisms for both pressure and velocity remain effective at the time asymptotic regime. Furthermore, the VMS algorithm that is analyzed also exhibits a compact global attractor and the corresponding absorbing sets, the improvement being the existence of an absorbing set for the stabilizing sub-grid component.

The outline of the article is as follows. In Section 1, we state the continuous problem and the basic results that describe its long-term behavior. In Section 2 we consider the semi-discrete in space finite element Galerkin approximation and how to stabilize it using our favored VMS sub-grid model. In Section 3, we prove the existence of an absorbing set in $L^2(\Omega)$, with particular emphasis on the new bounds due to stabilization. Finally, in Section 4, we prove the existence of an absorbing set in $H^1(\Omega)$ and a numerical global attractor in the two-dimensional case.

2. Problem statement.

2.1. Notation. Let Ω be any open set of \mathbb{R}^d , $d = 2$ or 3 . As usual $L^p(\Omega)$ (or $L^\infty(\Omega)$, respectively) denotes the space of real-valued functions defined on Ω with the p th-power integrable (or essentially bounded real-valued functions) for the Lebesgue measure. This space is a Banach space endowed with the norm $\|v\|_{L^p(\Omega)} = (\int_\Omega |v(\mathbf{x})|^p d\mathbf{x})^{1/p}$ (or $\|v\|_{L^\infty(\Omega)} = \text{ess sup}_{\mathbf{x} \in \Omega} |v(\mathbf{x})|$, respectively). In the particular case $p = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int_\Omega u(\mathbf{x})v(\mathbf{x})d\mathbf{x}.$$

On the other hand, $L^p_{\text{loc}}(\Omega)$ contains all the real-valued functions defined on Ω which belong to $L^p(\omega)$ for any compact subset ω of the open set Ω .

For m a non-negative integer and $p \geq 1$, we define the classical Sobolev spaces as

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega); \partial^k v \in L^p(\Omega) \forall |k| \leq m\},$$

associated to the norm

$$\|v\|_{W^{m,p}(\Omega)} = \left[\sum_{0 \leq |k| \leq m} \|\partial^k v\|_{L^p(\Omega)}^p \right]^{\frac{1}{p}},$$

where k is a multi-index; we will write this norm in compact form as $\|\cdot\|_{m,p}$. In the particular case $p = 2$, $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$, which is a Hilbert space with the obvious inner product and its associated norm $\|\cdot\|_m$. We will use bold-face letter for spaces of vector functions.

Let $\mathcal{C}_0^\infty(\Omega)$ be the space of infinitely differentiable functions with compact support in Ω . We denote by $\mathcal{D}(\Omega)$ the topological space of test functions in Ω . Its dual space, the space of distributions, is denoted by $\mathcal{D}'(\Omega)$. The closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ is defined by $W_0^{m,p}(\Omega)$ (analogously, $H_0^m(\Omega)$ when $p = 2$). The dual space of $W_0^{m,p}(\Omega)$ is identified by $W^{-m,q}(\Omega)$, q being the dual index to p , i.e., $\frac{1}{q} + \frac{1}{p} = 1$; analogously, we define H^{-m} as the dual space of $H^m(\Omega)$. In general, duality pairings will be indicated by the symbol $\langle \cdot, \cdot \rangle$.

Let $-\infty \leq a < b \leq +\infty$ and let X be a Banach space. Then $L^p(a, b; X)$ denotes the space of X -valued function on (a, b) such that $\int_a^b \|f(s)\|_X^p ds < \infty$ for $1 \leq p < \infty$ or $\text{ess sup}_{s \in (a,b)} \|f(s)\|_X < \infty$ for $p = \infty$. $C([a, b]; X)$ is the space of continuous X -valued functions such that $\sup_{t \in [a,b]} \|f(t)\|_X < \infty$. Analogously, $\mathcal{D}'(a, b; X)$ is the space of functions such that their X -norm have a distributional sense in (a, b) .

We now introduce the standard spaces of the Navier-Stokes framework. First of all, we define

$$\mathbf{J} = \{\mathbf{v} \in (\mathcal{C}_0^\infty(\Omega))^d; \nabla \cdot \mathbf{v} = 0\}.$$

Then, \mathbf{H} is the closure of \mathbf{J} in $\mathbf{L}^2(\Omega)$, characterized by

$$\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2(\Omega); \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

and \mathbf{V} is the closure of \mathbf{J} in $\mathbf{H}_0^1(\Omega)$, characterized by

$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1(\Omega); \nabla \cdot \mathbf{u} = 0, \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\}.$$

Finally, $L^2(\Omega)/\mathbb{R}$ is the quotient space of $L^2(\Omega)$ functions up to a constant with the norm $\|p\|_{L^2(\Omega)/\mathbb{R}} = \inf_{c \in \mathbb{R}} \|p + c\| = \|p - \int_\Omega p(\mathbf{x}) d\mathbf{x}\|$.

In what follows, C denotes a positive constant independent of the physical parameters. When dealing with the finite element problem, C will be independent also of the mesh size h . The value of C may be different at different occurrences. We will use the notation $A \gtrsim B$ and $A \lesssim B$ to indicate that $A \geq CB$ and $A \leq CB$, respectively, where A and B are expressions depending on functions that in the discrete case may depend on h as well.

2.2. The continuous problem. Let Ω be a bounded, open set of \mathbb{R}^d , $d = 2$ or 3 . Denote by $[0, \infty)$ the time interval, $Q = \Omega \times (0, \infty)$ the cylindrical space-time domain, and $\Sigma = \partial\Omega \times (0, \infty)$ its boundary. The flow of a viscous, incompressible, Newtonian fluid is described by the Navier-Stokes equations:

$$(2.1a) \quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } Q, \\ \nabla \cdot \mathbf{u} &= 0 & \text{in } Q. \end{cases}$$

The unknowns are the fluid velocity $\mathbf{u}(\mathbf{x}, t) : Q \rightarrow \mathbb{R}^d$ and the fluid pressure $p(\mathbf{x}, t) : Q \rightarrow \mathbb{R}$. The physical parameter $\nu > 0$ is the kinematic viscosity and \mathbf{f} is the external volume force applied to the fluid confined in Ω . These equations are supplemented with the initial condition

$$(2.1b) \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \quad \text{in } \Omega,$$

and the homogeneous Dirichlet boundary condition

$$(2.1c) \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{0} \quad \text{on } \Sigma.$$

We can also state the Navier-Stokes equations in weak or variational form. At almost every time value t , we seek for e.g. $[\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t)] \in L^2(0, T; \mathbf{H}_0^1(\Omega)) \times \mathcal{D}'(0, T; L^2(\Omega)/\mathbb{R})$ such that

$$(2.2a) \quad (\partial_t \mathbf{u}, \mathbf{v}) + \langle (\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v} \rangle + \nu (\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle,$$

$$(2.2b) \quad (q, \nabla \cdot \mathbf{u}) = 0,$$

for any $[\mathbf{v}, q] \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$, satisfying also the initial condition (2.1b) in a weak sense. The problem is posed with $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$ and force term $\mathbf{f} \in L^2(0, T; \mathbf{H}^{-1}(\Omega))$. Existence and uniqueness for (2.2) is an open problem in three dimensions. There are some partial results, like the existence of weak solutions; problem (2.2) has at least one weak solution that satisfies the following energy inequality (Leray inequality)

$$\frac{1}{2} \|\mathbf{u}(\mathbf{x}, t)\|^2 + \nu \int_0^t \|\nabla \mathbf{u}(\mathbf{x}, s)\|^2 ds \leq \frac{1}{2} \|\mathbf{u}(\mathbf{x}, 0)\|^2 + \int_0^t \langle \mathbf{f}(\mathbf{x}, s), \mathbf{u}(\mathbf{x}, s) \rangle ds$$

that implies

$$(2.3) \quad \frac{1}{2} \|\mathbf{u}(\mathbf{x}, t)\|^2 + \nu \int_0^t \|\nabla \mathbf{u}(\mathbf{x}, s)\|^2 ds \lesssim \frac{1}{\nu} \int_0^t \|\mathbf{f}\|_{-1}^2 ds + \|\mathbf{u}_h(\mathbf{x}, 0)\|^2.$$

Thus, $\mathbf{u} \in L^2(0, T; \mathbf{H}_0^1(\Omega)) \cap L^\infty(0, T; \mathbf{L}^2(\Omega))$ for all $0 < T < \infty$, under the regularity of the data indicated above.

The second term on the left-hand side of (2.3) represents the dissipation of kinetic energy. The larger scales of turbulent flows contain most of the kinetic energy of the system, that is transferred to smaller scales via the nonlinear term by an inertial and essentially inviscid mechanism. This process continuously creating smaller and smaller scales till forming $\mathcal{O}(\lambda_K)$ eddies, in which the viscous dissipation of energy finally takes place, i.e. $\nu \|\nabla \mathbf{u}(\mathbf{x}, s)\|^2$ becomes dominant. This process is known as the energy cascade.

Pressure stability can be obtained from the inf-sup condition

$$(2.4) \quad \inf_{q \in L^2(\Omega)} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(q, \nabla \cdot \mathbf{v})}{\|q\| \|\mathbf{v}\|_1} \geq \beta > 0,$$

which is a consequence of the surjectivity of the divergence operator from \mathbf{H}_0^1 to $L^2(\Omega)$ (see [31]). Even for the linear transient Stokes problem, in the most general setting in which the problem is well-posed, pressure stability in time is unclear (see [17]). Most of the mathematical analyses of the transient Navier-Stokes equations are obtained using divergence-free velocity spaces that allow to get rid of the pressure [36, 24, 25]. However, in some engineering applications pressure values are more important than fluid velocities, e.g. in fluid-structure interaction phenomena.

The previous results can be meaningless since the right-hand side of (2.3) can blow up as $t \rightarrow \infty$. Thus, new results have been obtained in order to understand the long-term behavior of (2.2). Let us assume that problem (2.1) is well posed for all $t \geq 0$ and \mathbf{f} is time-independent. We can describe this autonomous infinite-dimensional dynamical system by means of the semigroup $\{\mathcal{S}(t)\}_{t \geq 0}$, i.e. the family of operators:

$$\mathcal{S}(t) : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega), \quad \mathbf{u}_0(\mathbf{x}) \longrightarrow \mathbf{u}(\mathbf{x}, t), \quad t \geq 0.$$

The orbit associated to a given initial value is the set $\bigcup_{t \geq 0} \mathcal{S}(t) \mathbf{u}_0$. In dimension 2, it is known that the transient Navier-Stokes equations exhibit an absorbing set $\mathcal{B} \subset \mathbf{L}^2(\Omega)$, i.e.

for any $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$ there exists a time value $t_*(\mathbf{u}_0)$ such that $\bigcup_{t \geq t_*} \mathcal{S}(t)\mathbf{u}_0 \subset \mathcal{B}$ (see [37]). In fact, it is also possible to prove that there exists an absorbing set in $\mathbf{H}^1(\Omega)$. Due to the compactness of the $\mathbf{H}^1(\Omega)$ ball in $\mathbf{L}^2(\Omega)$, $\mathcal{S}(t)$ turns to be uniformly compact. In the asymptotic regime $t \rightarrow \infty$, it has been proved that all the orbits are attracted by a compact set \mathcal{A} of finite dimension, the global attractor [19, 37].

2.3. Finite element approximations. From now on, we assume that Ω is a subset of \mathbb{R}^d ($d = 2$ or 3) having a polygonal or polyhedral Lipschitz-continuous boundary, and $\{\mathcal{T}_h\}_{h>0}$ is a quasi-uniform family of triangulations of $\bar{\Omega}$, that is, $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$, with mesh size $h = \max_{K \in \mathcal{T}_h} h_K$, h_K being the diameter of the triangle K .

In order to get a conforming finite element approximation of the Navier-Stokes problem, we consider conforming finite element spaces $V_h \subset \mathbf{H}_0^1(\Omega)$ and $Q_h \subset L^2(\Omega)/\mathbb{R}$ for velocity and pressure respectively, with optimal interpolation properties. To simplify the exposition, we will consider $Q_h \subset \mathcal{C}^0(\Omega)$. Let us denote by $\Pi_{V_h}(\cdot)$ and $\Pi_{Q_h}(\cdot)$ the $L^2(\Omega)$ projections onto V_h and Q_h respectively, with optimal interpolation properties. We also denote by $\Pi_{V_h}^\perp(\cdot) := \text{Id}(\cdot) - \Pi_{V_h}(\cdot)$, the projection onto V_h^\perp , the space $L^2(\Omega)$ -orthogonal with respect to V_h . Then, the semi-discrete problem in space consists in finding e.g. $[\mathbf{u}_h, p_h] \in L^2(0, T; V_h) \times \mathcal{D}'(0, T; Q_h)$ such that

$$(2.5) \quad (\partial_t \mathbf{u}_h, \mathbf{v}_h) + \langle (\mathbf{u}_h \cdot \nabla) \mathbf{u}_h, \mathbf{v}_h \rangle + \nu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle,$$

$$(2.6) \quad (q_h, \nabla \cdot \mathbf{u}_h) = 0,$$

almost everywhere in time. Analogously to the continuous problem, it is easy to prove that the semi-discrete system (2.5) satisfies

$$\|\mathbf{u}_h(\mathbf{x}, t)\|^2 + \nu \int_0^t \|\nabla \mathbf{u}_h(\mathbf{x}, s)\|^2 ds \lesssim \frac{1}{\nu} \int_0^t \|\mathbf{f}\|_{-1}^2 ds + \|\mathbf{u}_h(\mathbf{x}, 0)\|^2.$$

Even for high Re , the viscous dissipative term of the continuous problem in (2.1) becomes dominant at the smallest scales of the flow; viscous effects extract energy to the system at the smallest scales, “killing” any fluctuation under a certain level, the Kolmogorov microscale λ_K (see [32, 34]). λ_K is obviously related to the number of nodes that are needed in a DNS computational mesh, since all the scales of the flow must be captured in such computations. When the computational mesh is substantially coarser than a DNS one, the smallest scales are $\mathcal{O}(h) \gg \lambda_K$, i.e. they belong to the inertial range. On the other hand, following the energy cascade, the energy from larger scales is transferred to the smallest scales. Since eddies in the range $\mathcal{O}(h)$ are much larger than the dissipative eddies that exist at Kolmogorov scales, kinetic energy is essentially not dissipated in this range. The viscous dissipation term $\nu \|\nabla \mathbf{u}_h\|^2$ never becomes important and, as a result, the smallest scales exhibit an energy pile-up (see [23]), leading to space instabilities.

Pressure stability for the Galerkin approximation of the Navier-Stokes equations cannot be attained from energy bounds. In order to mimic the mathematical structure of the continuous problem, we can build velocity-pressure finite element spaces satisfying a discrete inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|q_h\| \|\mathbf{v}_h\|_1} \geq \beta^* > 0,$$

where β^* is uniform with respect to h . Obviously, the discrete inf-sup condition is not a direct consequence of (2.4). In fact, some interesting velocity-pressure pairs, like equal-order velocity pressure approximations, fail to satisfy this condition, leading to pressure instabilities.

Using VMS stabilized finite element approximations, we get numerical methods with enhanced stability properties for which there is the hope that they can act as turbulence models. Pressure stability does not rely on a discrete inf-sup condition and fluid velocity bounds remain effective at high Re for mesh sizes $h \gg \lambda_K$, placed in the inertial range. Furthermore, the effect of the unresolved scales, i.e. scales in the range $(h, \lambda_K]$, into the captured scales is properly modeled; in particular, the viscous dissipation that takes place at the smallest unresolved scales. In fact, it has been proved that the energy spectra of VMS-based algorithms approximate accurately the continuous spectra till $\mathcal{O}(h)$ scales (see [22, 14, 35, 3]).

We do not include here the motivation of these algorithms, that can be found elsewhere (see [28, 30]). In particular, we consider the sub-grid scales to be orthogonal to the finite element velocity space and dynamic. In order to state the problem, we introduce the sub-grid velocity component $\tilde{\mathbf{u}}$, which is modeled. We assume the sub-grid pressure $\tilde{p} = 0$, since the terms obtained from this component are not essential for the good performance of the algorithm (see e.g. [9]). The sub-grid velocity belongs to the sub-grid space \tilde{V} , to be defined. The finite element approximation of the Navier-Stokes equations using a VMS dynamic orthogonal sub-grid model reads as follows: find $\mathbf{u}_h \in L^2_{\text{loc}}(0, \infty; V_h)$, $p_h \in L^2_{\text{loc}}(0, \infty; Q_h)$, and $\tilde{\mathbf{u}} \in L^2_{\text{loc}}(0, \infty; \tilde{V})$ such that

$$(2.7a) \quad (\partial_t \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \nu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)$$

$$(2.7b) \quad - (p_h, \nabla \cdot \mathbf{v}_h) - b(\mathbf{u}_h, \mathbf{v}_h, \tilde{\mathbf{u}}) = \langle \mathbf{f}, \mathbf{v}_h \rangle,$$

$$(2.7c) \quad (q_h, \nabla \cdot \mathbf{u}_h) - (\tilde{\mathbf{u}}, \nabla q_h) = 0,$$

$$(2.7d) \quad (\partial_t \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + \tau^{-1} (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \langle \mathbf{f}, \tilde{\mathbf{v}} \rangle - b(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{v}}) - (\nabla p_h, \tilde{\mathbf{v}}),$$

and

$$(2.8) \quad \mathbf{u}_h(0) = \mathbf{u}_{0h}, \quad \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0.$$

A proper initialization of the problem is obtained by using \mathbf{u}_{0h} and $\tilde{\mathbf{u}}_0$ solution of the projection problem

$$\begin{aligned} (\mathbf{u}_{0h}, \mathbf{v}_h) - (\xi_h, \nabla \cdot \mathbf{v}_h) &= (\mathbf{u}_0, \mathbf{v}_h), \\ (\nabla \cdot \mathbf{u}_{0h}, q_h) - (q_h, \tilde{\mathbf{u}}_0) &= 0, \\ (\tilde{\mathbf{u}}_0, \tilde{\mathbf{v}}) + (\nabla \xi_h, \tilde{\mathbf{v}}) &= (\mathbf{u}_0, \tilde{\mathbf{v}}). \end{aligned}$$

The nice feature of this choice is the fact that the initial velocity components satisfy the stabilized mass conservation equation, which can have important effects on the stability of the fully discrete problem (see [7]).

The so-called stabilization parameter is

$$(2.9) \quad \tau = \left(\frac{C_s \nu}{h^2} + \frac{C_c \|\mathbf{u}_h\|_{0,\ell}}{h|\Omega|^{\frac{1}{\ell}}} \right)^{-1}.$$

C_s are C_c are algorithmic constants independent of physical and numerical parameters that are usually motivated from the analysis of one-dimensional tests (see e.g. [9]). For practical purposes, a non-constant $\tau(\mathbf{x})$ is usually implemented, in which the global velocity norm is replaced by its pointwise modulus. The use of a variable stabilization parameter introduces some technical complications in the numerical analysis that have been faced in [11] for the linearized Oseen problem.

In the following, we assume that $2 \leq \ell \leq \infty$. Furthermore, we use the skew-symmetric form of the convective trilinear form (see [36])

$$b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}) = \langle (\mathbf{u}_h \cdot \nabla) \mathbf{v}_h, \mathbf{w} \rangle + \frac{1}{2} \langle \nabla \cdot \mathbf{u}_h, \mathbf{v}_h \cdot \mathbf{w} \rangle.$$

For the sake of conciseness in the following exposition, let us introduce the operator

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) : V_h \times V_h \longrightarrow \mathbf{L}^1(\Omega), \quad \mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{u}_h \cdot \nabla) \mathbf{v}_h + \frac{1}{2} (\nabla \cdot \mathbf{u}_h) \mathbf{v}_h.$$

Note that $\mathbf{L}^1(\Omega)$ is the space where \mathcal{N} can be shown to be bounded uniformly in h .

REMARK 2.1. *The $\tilde{\mathbf{u}}$ dependent term on the left-hand side of (2.7a) and (2.7c) stands for the effect of the sub-grid scales on the finite element component. The first one gives enhanced velocity stability whereas the second one provides pressure stability, as we shall see.*

REMARK 2.2. *The model for the sub-grid scale is (2.7d). τ comes from an algebraic approximation of the differential operator $-\nu \Delta(\cdot) + \mathcal{N}(\mathbf{u}_h, \cdot)$ that can be motivated by Fourier analysis. A key aspect of the previous formulation is the space for the subscales. In particular, we consider $\tilde{V} = (\text{Im}(\mathcal{L}) \cup \{\mathbf{f}\}) \cap V_h^\perp$, where*

$$\mathcal{L}(\mathbf{v}_h, \mathbf{w}_h, q_h) : V_h \times V_h \times Q_h \longrightarrow \mathbf{L}^2(\Omega), \quad \mathcal{L}(\mathbf{v}_h, \mathbf{w}_h, q_h) = \mathcal{N}(\mathbf{v}_h, \mathbf{w}_h) + \nabla q_h.$$

Note now that $\mathbf{L}^2(\Omega)$ is the space where \mathcal{L} can be shown to be bounded uniformly in h . This implies that (2.7d) can be understood in a pointwise setting as:

$$(2.10) \quad \partial_t \tilde{\mathbf{u}} + \tau^{-1} \tilde{\mathbf{u}} = \Pi_{V_h}^\perp (\mathbf{f} - \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h) - \nabla p_h).$$

This makes the subscale model very cheap, since it is a local problem at every finite element of the triangulation. In its numerical implementation, the sub-grid component will be simply evaluated by using (2.10) at every integration point of every finite element.

REMARK 2.3. *In general, the finite element residual in the right hand side of (2.7d) must include the viscous term in order to have a consistent method, i.e., it must be:*

$$\langle \mathbf{f}, \tilde{\mathbf{v}} \rangle - b(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{v}}) - (\nabla p_h, \tilde{\mathbf{v}}) + \sum_{K \in \mathcal{T}_h} (\nu \Delta \mathbf{u}_h, \tilde{\mathbf{v}})_K.$$

The subscript K in the last term indicates that the Laplacian is considered inside every finite element separately. It is obvious that the viscous term vanishes for piecewise linear approximations. However, for higher order polynomial approximations, this term should be kept. In the following, we perform the analysis omitting the viscous term for the sake of clarity (it would be crucial to prove convergence, but we will restrict ourselves to analyze stability). The introduction of this term is however straightforward, simply using an inverse inequality, under some assumptions over the constant C_s . We refer to [2] for more details. Let us also mention that there is a slight modification of the method that allows us to avoid the need for introducing the viscous term in the residual, which consists in replacing $\Pi_{V_h}^\perp$ in (2.10) by the projection orthogonal to the velocity finite element space without boundary conditions. This leads to an optimal order consistency error, although full stability is only proved under a technical (and very mild) condition on the finite element mesh (see [12, 11]).

The pointwise and weak sub-grid equations, (2.10) and (2.7d) respectively, are equivalent, but we will use the weak formulation for the subsequent analysis. This weak form of the sub-grid model is not standard. We refer to [2, 13, 14] for stability and convergence analyses for dynamic orthogonal sub-grid models applied to linear problems, namely convection-diffusion-reaction systems and the Stokes problem. The linearized stationary problem is fully

analyzed in [11]. In the next sections we will analyze the stability of this nonlinear finite-dimensional problem (2.7), with special emphasis on its long-term behavior.

In the next lemma, we prove existence and uniqueness for system (2.7).

LEMMA 2.1. *The semi-discrete problem (2.7) has a unique solution such that*

$$\mathbf{u}_h \in H_{\text{loc}}^1(0, \infty; V_h), \quad \tilde{\mathbf{u}} \in H_{\text{loc}}^1(0, \infty; \tilde{V}), \quad p_h \in L_{\text{loc}}^2(0, \infty; Q_h).$$

Proof. Let us choose a finite final time $T^* < \infty$ and discretize (2.7) in time by using a linear backward-Euler scheme. Let us assume a uniform partition $0 = t_0 < t_1 < \dots < t_N = T^*$ of $[0, T]$ with $k = \frac{T}{N}$ the time step size. The problem consists in, given $(\mathbf{u}_h^n, \tilde{\mathbf{u}}^n) \in V_h \times \tilde{V}$, seek $(\mathbf{u}^{n+1}, \tilde{\mathbf{u}}^{n+1}, p_h^{n+1})$ satisfying

$$(2.11a) \quad (\delta_t \mathbf{u}_h^{n+1}, \mathbf{v}_h) + b(\mathbf{u}_h^n, \mathbf{u}_h^{n+1}, \mathbf{v}_h) + \nu (\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) - b(\mathbf{u}_h^n, \mathbf{v}_h, \tilde{\mathbf{u}}^{n+1}) = \langle \mathbf{f}, \mathbf{v}_h \rangle,$$

$$(2.11b) \quad (q_h, \nabla \cdot \mathbf{u}_h^{n+1}) - (\tilde{\mathbf{u}}^{n+1}, \nabla q_h) = 0,$$

$$(2.11c) \quad (\delta_t \tilde{\mathbf{u}}^{n+1}, \tilde{\mathbf{v}}) + \tau_n^{-1} (\tilde{\mathbf{u}}^{n+1}, \tilde{\mathbf{v}}) + b(\mathbf{u}_h^n, \mathbf{u}_h^{n+1}, \tilde{\mathbf{v}}) + (\nabla p_h^{n+1}, \tilde{\mathbf{v}}) = \langle \mathbf{f}, \tilde{\mathbf{v}} \rangle,$$

where

$$\delta_t \mathbf{u}_h^{n+1} := \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{k} \quad \text{and} \quad \tau_n = \left(\frac{C_s \nu}{h^2} + \frac{C_c \|\mathbf{u}_h^n\|_{0,\ell}}{h|\Omega|^{\frac{1}{\ell}}}} \right)^{-1}.$$

Since (2.11) is a square system existence is equivalent to uniqueness. Furthermore, uniqueness is straightforwardly deduced from the weak estimates for (2.11). These bounds are naturally obtained by selecting $\mathbf{v}_h = \mathbf{u}_h^{n+1}$, $q_h = p_h^{n+1}$, and $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}^{n+1}$ in (2.11). This way, one finds

$$\delta_t (\|\mathbf{u}_h^{n+1}\|^2 + \|\tilde{\mathbf{u}}^{n+1}\|^2) + \nu \|\nabla \mathbf{u}_h^{n+1}\|^2 + \tau_n^{-1} \|\tilde{\mathbf{u}}^{n+1}\|^2 \leq C_P \frac{|\Omega|^{\frac{2}{d}}}{2\nu} \|\mathbf{f}\|^2,$$

where we have used the Poincaré inequality, C_P being its corresponding constant. From this inequality, analogous estimates to those in Theorem 3.1 are obtained.

Now, testing $\mathbf{v}_h = k\delta_t \mathbf{u}_h^{n+1}$, $\tilde{\mathbf{v}} = k\delta_t \tilde{\mathbf{u}}^{n+1}$ in (2.11), we get

$$\begin{aligned} & k \|\delta_t \mathbf{u}_h^{n+1}\|^2 + k \|\delta_t \tilde{\mathbf{u}}^{n+1}\|^2 + \delta_t \left(\frac{\nu}{2} \|\nabla \mathbf{u}_h^{n+1}\|^2 + \frac{C_s \nu}{h^2} \|\tilde{\mathbf{u}}^{n+1}\|^2 \right) \\ &= -k b(\mathbf{u}_h^n, \mathbf{u}_h^{n+1}, \delta_t \mathbf{u}_h^{n+1}) + k b(\mathbf{u}_h^n, \delta_t \mathbf{u}_h^{n+1}, \tilde{\mathbf{u}}^{n+1}) - k b(\mathbf{u}_h^n, \mathbf{u}_h^{n+1}, \delta_t \tilde{\mathbf{u}}^{n+1}) \\ &\quad - k \frac{C_c \|\mathbf{u}_h\|_\ell}{h} (\tilde{\mathbf{u}}^{n+1}, \delta_t \tilde{\mathbf{u}}^{n+1}) + k(\mathbf{f}, \delta_t \mathbf{u}_h^{n+1} + \delta_t \tilde{\mathbf{u}}^{n+1}). \end{aligned}$$

The pressure terms vanish due to the fact that

$$(q_h, \nabla \cdot \delta_t \mathbf{u}_h^{n+1}) - (\delta_t \tilde{\mathbf{u}}^{n+1}, \nabla q_h) = 0$$

holds for a proper initialization. Thus, it is easy to find that there is a positive constant $C(h, \nu, T^*)$ such that

$$\begin{aligned} & k \|\delta_t \mathbf{u}_h^{n+1}\|^2 + k \|\delta_t \tilde{\mathbf{u}}^{n+1}\|^2 + \delta_t \left(\frac{\nu}{2} \|\nabla \mathbf{u}_h^{n+1}\|^2 + \frac{C_s \nu}{h^2} \|\tilde{\mathbf{u}}^{n+1}\|^2 \right) \\ &\leq C(h, \nu, T^*) k (\nu \|\nabla \mathbf{u}_h^{n+1}\|^2 + \tau^{-1} \|\tilde{\mathbf{u}}^{n+1}\|^2) \end{aligned}$$

by taking into account inverse inequalities. At this level, the fact that the bounds depend on h , ν , and T^* is irrelevant. From here, estimates for the velocity time derivative can be obtained.

Finally, we establish an estimate for the pressure. Take $\mathbf{v}_h = k \Pi_h(\nabla p_h)$ and $\tilde{\mathbf{v}} = k \Pi_h^\perp(\nabla p_h)$ in (2.11) to get

$$\begin{aligned} k \|\nabla p_h^{n+1}\|^2 &= k \|\Pi_h(\nabla p_h^{n+1})\|^2 + k \|\Pi_h^\perp(\nabla p_h^{n+1})\|^2 = -k(\delta_t \mathbf{u}_h^{n+1}, \Pi_h(\nabla p_h^{n+1})) \\ &\quad - k(\delta_t \tilde{\mathbf{u}}^{n+1}, \Pi_h^\perp(\nabla p_h^{n+1})) - k b(\mathbf{u}_h^n, \mathbf{u}_h^{n+1}, \nabla p_h^{n+1}) + k b(\mathbf{u}_h^n, \Pi_h(\nabla p_h^{n+1}), \tilde{\mathbf{u}}^{n+1}) \\ &\quad - k \nu(\nabla \mathbf{u}^{n+1}, \nabla \Pi_h(\nabla p_h^{n+1})) - k \tau^{-1}(\tilde{\mathbf{u}}^{n+1}, \Pi_h^\perp(\nabla p_h^{n+1})) + k(\mathbf{f}, \nabla p_h^{n+1}). \end{aligned}$$

Again inverse inequalities and the previous bounds provide an estimate for the pressure. A compactness result may be established using the Aubin-Lions theorem. Combining the above information, the passage to the limit gives us the existence of a solution with the desired regularity. Uniqueness is easily proved by making a comparison between two different solutions. \square

Let us prove some preliminary results that will be needed in the following sections. First, we analyze the approximation properties of the VMS stabilized finite element approximation of the steady Stokes problem using orthogonal subscales. The Stokes problem reads as: find $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ and $p \in L_0^2(\Omega)/\mathbb{R} \cap H^1(\Omega)$ such that

$$\begin{aligned} (2.12a) \quad & -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} && \text{in } \Omega, \\ (2.12b) \quad & \nabla \cdot \mathbf{u} = 0 && \text{in } \Omega. \end{aligned}$$

for any $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Let us assume in the following that the solution of system (2.12) satisfies the elliptic regularity assumption

$$(2.13) \quad \nu \|\mathbf{u}\|_2 + \|p\|_1 \leq \|\mathbf{f}\|,$$

which is known to be true when Ω satisfies some regularity properties (see e.g. [21]).

The stabilized finite element approximation of the Stokes problem, using orthogonal subscales, reads as (see [2]): find $\mathbf{u}_h \in V_h$, $p_h \in Q_h$ and $\tilde{\mathbf{u}} \in \tilde{V}$ such that

$$(2.14a) \quad \nu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle,$$

$$(2.14b) \quad (q_h, \nabla \cdot \mathbf{u}_h) - (\tilde{\mathbf{u}}, \nabla q_h) = 0,$$

$$(2.14c) \quad \tau_\nu^{-1}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + (\nabla p_h, \tilde{\mathbf{v}}) = \langle \mathbf{f}, \tilde{\mathbf{v}} \rangle,$$

where $\tau_\nu := \frac{h^2}{\nu}$ and \tilde{V} is designed as above, but now using $\mathcal{L} : Q_h \rightarrow \mathbf{L}^2(\Omega)$ with $\mathcal{L}(q_h) = \nabla q_h$.

LEMMA 2.2 (Error estimates for (2.14)). *Let us assume that the elliptic regularity assumptions (2.13) hold. Then, the solution $(\mathbf{u}_h, p_h, \tilde{\mathbf{u}})$ of problem (2.14) and the continuous solution (\mathbf{u}, p) of problem (2.12) satisfy the error estimates:*

$$(2.15) \quad \nu^{\frac{1}{2}} \|\nabla(\mathbf{u} - \mathbf{u}_h)\| + \frac{\nu^{\frac{1}{2}}}{h} \|\tilde{\mathbf{u}}\| + \frac{1}{\nu^{\frac{1}{2}}} \|p - p_h\| \lesssim \frac{h}{\nu^{\frac{1}{2}}} \|\mathbf{f}\|.$$

Proof. We indicate the finite element component of the error functions with

$$\mathbf{e}_h = \Pi_{V_h}(\mathbf{u}) - \mathbf{u}_h, \quad \psi_h = \Pi_{Q_h}(p) - p_h.$$

Subtracting the weak form of system (2.12) and (2.14), we obtain the error system

$$(2.16a) \quad \nu (\nabla \mathbf{e}_h, \nabla \mathbf{v}_h) - (\psi_h, \nabla \cdot \mathbf{v}_h) = \langle \mathcal{E}^1, \mathbf{v}_h \rangle,$$

$$(2.16b) \quad (\nabla \cdot \mathbf{e}_h, q_h) + (\tilde{\mathbf{u}}, \nabla q_h) = \langle \mathcal{E}^2, q_h \rangle,$$

with

$$\begin{aligned}\langle \mathcal{E}^1, \mathbf{v}_h \rangle &:= -\nu (\nabla(\mathbf{u} - \Pi_{V_h}(\mathbf{u})), \nabla \mathbf{v}_h) + (p - \Pi_{Q_h}(p), \nabla \cdot \mathbf{v}_h), \\ \langle \mathcal{E}^2, q_h \rangle &:= -(\nabla \cdot (\mathbf{u} - \Pi_{V_h}(\mathbf{u})), q_h).\end{aligned}$$

Let us re-write the subscale equation as follows

$$(2.17) \quad \frac{\nu}{h^2} (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) - (\nabla(\Pi_{Q_h}(p) - p_h), \tilde{\mathbf{v}}) = (\mathbf{f} - \nabla \Pi_{Q_h}(p), \tilde{\mathbf{v}}) =: \langle \mathcal{E}^3, \tilde{\mathbf{v}} \rangle$$

We denote by $\varepsilon_i(v) := |v - \Pi_{V_h}(v)|_i$, where $|\cdot|_i$ denotes the semi-norm in $H^i(\Omega)$. We can easily bound the right-hand side of system (2.16a) using integration by parts and invoking the momentum equation in (2.12) as follows:

$$\begin{aligned}\langle \mathcal{E}^1, \mathbf{v}_h \rangle &\lesssim \left(\nu^{\frac{1}{2}} \varepsilon_1(\mathbf{u}) + \nu^{-\frac{1}{2}} \varepsilon_0(p) \right) \nu^{\frac{1}{2}} \|\nabla \mathbf{v}_h\|, \\ \langle \mathcal{E}^2, q_h \rangle &\lesssim \nu^{\frac{1}{2}} h^{-1} \varepsilon_0(\mathbf{u}) \nu^{-\frac{1}{2}} h \|\Pi_{V_h}^\perp(\nabla q_h)\|, \\ \langle \mathcal{E}^3, \tilde{\mathbf{v}} \rangle &\lesssim h \nu^{-\frac{1}{2}} (\nu \|\Delta \mathbf{u}\| + \|\nabla(p - \Pi_{Q_h}(p))\|) \nu^{\frac{1}{2}} h^{-1} \|\tilde{\mathbf{v}}\|.\end{aligned}$$

Let us define the interpolation and consistency error function

$$E(h) := \nu^{\frac{1}{2}} h^{-1} \varepsilon_0(\mathbf{u}) + \nu^{\frac{1}{2}} \varepsilon_1(\mathbf{u}) + h \nu^{\frac{1}{2}} \|\Delta \mathbf{u}\| + \nu^{-\frac{1}{2}} \varepsilon_0(p) + h \nu^{-\frac{1}{2}} \varepsilon_1(p).$$

Now, we take $\mathbf{v}_h = \mathbf{e}_h$, $q_h = \psi_h$ and $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}$ in (2.16)-(2.17) respectively. We obtain:

$$(2.18) \quad \nu \|\nabla \mathbf{e}_h\|^2 + \frac{\nu}{h^2} \|\tilde{\mathbf{u}}\|^2 \lesssim E(h) \left(\nu^{\frac{1}{2}} \|\nabla \mathbf{e}_h\| + \frac{h}{\nu^{\frac{1}{2}}} \|\Pi_{V_h}^\perp \nabla \psi_h\| + \frac{\nu^{\frac{1}{2}}}{h} \|\tilde{\mathbf{u}}\| \right).$$

We can find a bound for $h \nu^{-\frac{1}{2}} \|\Pi_{V_h}^\perp \nabla \psi_h\|$ using the subscale equation (2.14c) in its pointwise sense and (2.12a), getting

$$(2.19) \quad \|\Pi_{V_h}^\perp(\nabla \psi_h)\| \lesssim \frac{\nu}{h^2} \|\tilde{\mathbf{u}}\| + \|\mathbf{f} - \nabla p\| + \|\nabla(p - \Pi_{Q_h}(p))\| \lesssim \frac{\nu}{h^2} \|\tilde{\mathbf{u}}\| + \nu \|\Delta \mathbf{u}\| + \|\nabla p\|,$$

where we have used the $H^1(\Omega)$ -stability of $\Pi_{V_h}(\cdot)$ for quasi-uniform meshes (see [4]). This expression is now incorporated into (2.18) to get

$$\nu \|\nabla \mathbf{e}_h\|^2 + \frac{\nu}{h^2} \|\tilde{\mathbf{u}}\|^2 \lesssim E(h) \left(\nu^{\frac{1}{2}} \|\nabla \mathbf{e}_h\| + h \nu^{\frac{1}{2}} \|\Delta \mathbf{u}\| + \frac{h}{\nu^{\frac{1}{2}}} \|\nabla p\| + \frac{\nu^{\frac{1}{2}}}{h} \|\tilde{\mathbf{u}}\| \right).$$

The regularity assumptions in the theorem allow to obtain $\nu \|\mathbf{u}\|_2 + \|p\|_1 \leq \|\mathbf{f}\|$. It leads to

$$\nu^{\frac{1}{2}} \|\nabla \mathbf{e}_h\| + h^{-1} \nu^{\frac{1}{2}} \|\tilde{\mathbf{u}}\| \lesssim h \nu^{-\frac{1}{2}} \|\mathbf{f}\|,$$

where we have used the fact that $E(h) \lesssim h \nu^{-\frac{1}{2}} \|\mathbf{f}\|$, a direct consequence of classical interpolation theory. Global errors (2.15) are obtained using standard interpolation results and the triangle inequality.

In order to get stability bounds over the pressure, we test (2.16a) with $\mathbf{v}_h = \Pi_{V_h}(\nabla p - \nabla p_h)$. Using an inverse inequality, we easily get

$$h \nu^{-\frac{1}{2}} \|\Pi_{V_h}(\nabla p - \nabla p_h)\| \lesssim \nu^{\frac{1}{2}} \|\nabla(\mathbf{u} - \mathbf{u}_h)\| \lesssim E(h).$$

This bound, together with (2.19) and the definition of $E(h)$, lead to $h\nu^{-\frac{1}{2}}\|\nabla p - \nabla p_h\| \lesssim E(h)$. On the other hand, for all $q \in L^2(\Omega)$, there exists $\mathbf{v}_q \in \mathbf{H}_0^1(\Omega)$ such that:

$$(q, \nabla \cdot \mathbf{v}_q) \gtrsim \|q\| \|\mathbf{v}_q\|_1,$$

due to (2.4). Therefore, for $Q_h \subset \mathcal{C}^0(\Omega)$ we can find \mathbf{v}_e such that:

$$\begin{aligned} \|p - p_h\| \|\mathbf{v}_e\|_1 &\lesssim (\nabla(p - p_h), \mathbf{v}_e) \\ &\leq (\nabla(p - p_h), \mathbf{v}_e - \Pi_{V_h}(\mathbf{v}_e)) + (\nabla(p - p_h), \Pi_{V_h}(\mathbf{v}_e)) \\ &\lesssim (\nabla(p - p_h), \mathbf{v}_e - \Pi_{V_h}(\mathbf{v}_e)) - \nu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla \Pi_{V_h}(\mathbf{v}_e)) \\ &\lesssim \|\nabla(p - p_h)\| h \|\mathbf{v}_e\|_1 + \nu \|\nabla(\mathbf{u} - \mathbf{u}_h)\| \|\mathbf{v}_e\|_1. \end{aligned}$$

We easily get $\nu^{-\frac{1}{2}}\|p - p_h\| \lesssim E(h)$. \square

Finally, let us prove a discrete version of a well-known interpolation inequality (see [1]) that will be required for the treatment of the nonlinear terms. Let us introduce the discrete Laplacian $\Delta_h \mathbf{u}_h \in V_h$, solution of

$$(\Delta_h \mathbf{u}_h, \mathbf{v}_h) = (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h.$$

LEMMA 2.3. *Let $\Omega \subset \mathbb{R}^2$ have the elliptic regularity assumptions (2.13) and consider a quasi-uniform family of finite element meshes. For any $\mathbf{u}_h \in V_h$, the following inequality holds:*

$$\|\nabla \mathbf{u}_h\|_{0,4} \lesssim \|\nabla \mathbf{u}_h\|^{1/2} \|\Delta_h \mathbf{u}_h\|^{1/2}.$$

Proof. Let us consider $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ such that $\Delta \mathbf{u} = \Delta_h \mathbf{u}_h$. Assuming regularity of the domain, e.g. a convex domain Ω , we get the classical error estimates:

$$(2.20) \quad \|\mathbf{u} - \mathbf{u}_h\| + h \|\nabla(\mathbf{u} - \mathbf{u}_h)\| \lesssim h^2 \|\Delta \mathbf{u}\|,$$

where the error estimate in the $L^2(\Omega)$ norm is proved using Aubin-Nitsche duality arguments (see e.g. [17]). In particular, we get the error estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,4} \lesssim h^{3/2} \|\Delta \mathbf{u}\|,$$

due to the Gagliardo-Nirenberg inequality (see [1]).

Using the inverse inequality $\|\nabla \mathbf{v}_h\|_{0,p} \lesssim h^{-1} \|\mathbf{v}_h\|_{0,p}$ (for $1 \leq p \leq \infty$) and the definition of Δ_h , we easily get

$$\|\Delta_h \mathbf{u}_h\| \lesssim h^{-1} \|\nabla \mathbf{u}_h\| \lesssim h^{-2} \|\mathbf{u}_h\|.$$

These inverse estimates, together with the error estimates (2.20) and the definition of \mathbf{u} , lead to

$$\|\mathbf{u}\| \lesssim \|\mathbf{u}_h\|, \quad \|\nabla \mathbf{u}\| \lesssim \|\nabla \mathbf{u}_h\|.$$

Let us introduce the Scott-Zang interpolation operator $\mathcal{SZ}_{V_h}(\cdot)$ with regard to V_h (see [17, 4]). Assuming the regularity of the Laplace operator, i.e. $\|\mathbf{u}\|_2 \lesssim \|\Delta \mathbf{u}\|$ for any $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$, and using the previous inequalities, we obtain:

$$\begin{aligned} \|\nabla \mathbf{u}_h\|_{0,4} &\leq \|\nabla(\mathcal{SZ}_{V_h}(\mathbf{u}) - \mathbf{u}_h)\|_{0,4} + \|\nabla \mathcal{SZ}_{V_h}(\mathbf{u})\|_{0,4} \\ &\lesssim h^{-1} \|\mathcal{SZ}_{V_h}(\mathbf{u}) - \mathbf{u}_h\|_{0,4} + \|\nabla \mathbf{u}\|_{0,4} \\ &\lesssim h^{-1} (\|\mathcal{SZ}_{V_h}(\mathbf{u}) - \mathbf{u}\|_{0,4} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4}) + \|\nabla \mathbf{u}\|_1^{\frac{1}{2}} \|\nabla \mathbf{u}\|_2^{\frac{1}{2}} \\ &\lesssim h^{\frac{1}{2}} \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}} \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}} + \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}} \|\nabla \mathbf{u}_h\|^{\frac{1}{2}} \\ &\lesssim \|\nabla \mathbf{u}_h\|^{\frac{1}{2}} \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}}. \end{aligned}$$

For the bound in the second line we have used the $W^{1,p}(\Omega)$ -stability of the Scott-Zang interpolation. Then, we have invoked a Gagliardo-Nirenberg inequality and (2.20) for the obtention of the bounds in the third and fourth line. \square

3. Long-term stability in $L^\infty(0, \infty; \mathbf{L}^2(\Omega))$. Our first result proves that the VMS finite element approximation of the Navier-Stokes equations (2.7) exhibits an absorbing set in $\mathbf{L}^2(\Omega)$. A key difference with respect to previous analysis is the proof of an $\mathbf{L}^2(\Omega)$ absorbing set for the sub-grid component too. We prove the existence of the $\mathbf{L}^2(\Omega)$ absorbing set and some long-term stability bounds in the next theorem that holds in 2 and 3 dimensions. When there is no confusion, we will omit the time label for the unknowns.

Let us start this section with short-term stability bounds that are straightforward from (3.2) below, for $T < \infty$.

THEOREM 3.1 (Short-term stability). *Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 . When the time domain is bounded, i.e. $T < \infty$, system (2.7) with $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$, $\mathbf{f} \in L^2(0, T; \mathbf{H}^{-1}(\Omega))$ satisfies the energy-type inequality*

$$\begin{aligned} & (\|\mathbf{u}_h(t)\|^2 + \|\tilde{\mathbf{u}}(t)\|^2) + \int_0^T (\nu \|\nabla \mathbf{u}_h\|^2 + \tau^{-1} \|\tilde{\mathbf{u}}\|^2) ds \\ & \lesssim \int_0^T \frac{1}{2\nu} \|\mathbf{f}\|_{-1}^2 ds + \|\mathbf{u}_h(0)\|^2 \quad \text{for all } t \in [0, T], \end{aligned}$$

which implies that

$$\begin{aligned} \mathbf{u}_h & \in L^\infty(0, T; \mathbf{L}^2(\Omega)), & \tilde{\mathbf{u}} & \in L^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \nabla \mathbf{u}_h & \in L^2(0, T; \mathbf{L}^2(\Omega)), & \tau^{-\frac{1}{2}} \tilde{\mathbf{u}} & \in L^2(0, T; \mathbf{L}^2(\Omega)). \end{aligned}$$

REMARK 3.1. *The previous stability results are obtained with the minimum requirement that the body force $\mathbf{f} \in L^2(0, T; \mathbf{H}^{-1}(\Omega))$. However, those stability results become meaningless as $T \rightarrow \infty$, since $\int_0^T \|\mathbf{f}\|_{-1}^2 ds$ blows-up even for a constant body force, e.g. the gravity force. In the next theorem, we will obtain long-term stability estimates that remain effective when $T \rightarrow \infty$. In order to obtain these results, a slightly more regular body force is needed, i.e. $\mathbf{f} \in L^\infty(0, \infty; \mathbf{L}^2(\Omega))$.*

Let us introduce the nondimensional number $G := \frac{|\Omega|^{\frac{2}{d}}}{\nu^2} \|\mathbf{f}\|_{L^\infty(0, \infty; \mathbf{L}^2(\Omega))}$, defined in [18] as the dimensionless Grashof number; G can also be interpreted as Re^2 . In the next theorems, we make use of $\rho := \nu G$

THEOREM 3.2 (Long-term stability in $L^\infty(0, \infty; \mathbf{L}^2(\Omega))$). *Let us assume that the elliptic regularity assumptions (2.13) hold. Then, the solution of problem (2.7) for $d = 2, 3$ satisfies*

$$\begin{aligned} \mathbf{u}_h & \in L^\infty(0, \infty; \mathbf{L}^2(\Omega)), & \tilde{\mathbf{u}} & \in L^\infty(0, \infty; \mathbf{L}^2(\Omega)), \\ \nabla \mathbf{u}_h & \in L_{\text{loc}}^2(0, \infty; \mathbf{L}^2(\Omega)), & \tau^{-\frac{1}{2}} \tilde{\mathbf{u}} & \in L_{\text{loc}}^2(0, \infty; \mathbf{L}^2(\Omega)), \end{aligned}$$

for $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$ and $\mathbf{f} \in L^\infty(0, \infty; \mathbf{L}^2(\Omega))$. On the other hand, the following inequality holds,

$$(3.1) \quad \limsup_{t \rightarrow \infty} (\|\mathbf{u}_h(t)\|^2 + \|\tilde{\mathbf{u}}(t)\|^2) \lesssim \frac{|\Omega|^{\frac{4}{d}}}{\nu^2} \|\mathbf{f}\|_{L^\infty(0, \infty; \mathbf{L}^2(\Omega))}^2.$$

which implies the existence of an absorbing set in $\mathbf{L}^2(\Omega)$.

Proof. In order to prove the theorem, we formally start taking $\mathbf{v}_h = \mathbf{u}_h$, $q_h = p_h$ and $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}$ in (2.7). In fact, system (2.7) does not have pointwise sense for $\mathbf{f} \in L^\infty(0, t; \mathbf{L}^2(\Omega))$, and so its solution. In this situation, using the density of $\mathcal{C}_0^\infty([0, t])$ in $L^\infty(0, t)$, we can always find $\mathbf{f}_\varepsilon \in \mathcal{C}_0^\infty([0, t]; \mathbf{L}^2(\Omega))$ such that $\|\mathbf{f} - \mathbf{f}_\varepsilon\|_{L^\infty(0, t; \mathbf{L}^2(\Omega))} < \varepsilon$. The perturbed problem has pointwise sense in time, we can test against the solution for every time value, obtain the desired stability estimates and take the vanishing ε limit. Using this procedure we get, invoking the fact that the convective form is skew-symmetric,

$$(3.2) \quad \frac{1}{2} \partial_t (\|\mathbf{u}_h\|^2 + \|\tilde{\mathbf{u}}\|^2) + \nu \|\nabla \mathbf{u}_h\|^2 + \tau^{-1} \|\tilde{\mathbf{u}}\|^2 = (\mathbf{f}, \mathbf{u}_h + \tilde{\mathbf{u}}).$$

In order to bound the right-hand side, we use Hölder and Poincaré inequalities, the fact that $h < |\Omega|^{\frac{1}{d}}$ and the expression for τ , obtaining:

$$(3.3) \quad (\mathbf{f}, \mathbf{u}_h + \tilde{\mathbf{u}}) \leq C_P \frac{|\Omega|^{\frac{2}{d}}}{2\nu} \|\mathbf{f}\|^2 + \frac{\nu}{2} \|\nabla \mathbf{u}_h\|^2 + \frac{\tau^{-1}}{2} \|\tilde{\mathbf{u}}\|^2.$$

Combining (3.2) and (3.3), we get:

$$(3.4) \quad \frac{1}{2} \partial_t (\|\mathbf{u}_h\|^2 + \|\tilde{\mathbf{u}}\|^2) + \frac{1}{2} \nu \|\nabla \mathbf{u}_h\|^2 + \frac{\tau^{-1}}{2} \|\tilde{\mathbf{u}}\|^2 \leq C_P \frac{|\Omega|^{\frac{2}{d}}}{2\nu} \|\mathbf{f}\|^2,$$

which, integrated over $[t_0, t]$, leads to

$$(3.5) \quad \begin{aligned} & (\|\mathbf{u}_h(t)\|^2 + \|\tilde{\mathbf{u}}(t)\|^2) + \int_{t_0}^t (\nu \|\nabla \mathbf{u}_h\|^2 + \tau^{-1} \|\tilde{\mathbf{u}}\|^2) \, ds \\ & \lesssim \int_{t_0}^t \frac{|\Omega|^{\frac{2}{d}}}{2\nu} \|\mathbf{f}\|^2 \, ds + (\|\mathbf{u}_h(t_0)\|^2 + \|\tilde{\mathbf{u}}(t_0)\|^2). \end{aligned}$$

On the other hand, using the Poincaré inequality in (3.4) and the expression for τ , we get:

$$\partial_t (\|\mathbf{u}_h\|^2 + \|\tilde{\mathbf{u}}\|^2) + \nu |\Omega|^{-\frac{2}{d}} (\|\mathbf{u}_h\|^2 + \|\tilde{\mathbf{u}}\|^2) \lesssim \frac{|\Omega|^{\frac{4}{d}}}{2\nu} \|\mathbf{f}\|^2.$$

Now, we can use the classical Gronwall lemma (see [36]), obtaining:

$$\begin{aligned} (\|\mathbf{u}_h(t)\|^2 + \|\tilde{\mathbf{u}}(t)\|^2) & \lesssim \left(1 - \exp\left(-\nu |\Omega|^{-\frac{2}{d}} t\right)\right) \frac{|\Omega|^{\frac{4}{d}}}{\nu^2} \|\mathbf{f}\|_{L^\infty(0, t; \mathbf{L}^2(\Omega))}^2 \\ & \quad + \exp\left(-\nu |\Omega|^{-\frac{2}{d}} t\right) (\|\mathbf{u}_h(0)\|^2 + \|\tilde{\mathbf{u}}(0)\|^2). \end{aligned}$$

The previous inequality proves the $L^\infty(0, \infty; \mathbf{L}^2(\Omega))$ stability results and the existence of the $\mathbf{L}^2(\Omega)$ absorbing set, such that the orbit associated to any $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$ enters this subset at some time $t^*(\rho, \mathbf{u}_0)$. Now, taking the limit superior for $t \rightarrow \infty$, we get

$$\limsup_{t \rightarrow \infty} (\|\mathbf{u}_h(t)\|^2 + \|\tilde{\mathbf{u}}(t)\|^2) \lesssim \frac{|\Omega|^{\frac{4}{d}}}{\nu^2} \|\mathbf{f}\|_{L^\infty(0, t; \mathbf{L}^2(\Omega))}^2.$$

This proves the second part of the theorem. On the other hand, we get from (3.5) that

$$\int_{t_0}^t (\nu \|\nabla \mathbf{u}_h\|^2 + \tau^{-1} \|\tilde{\mathbf{u}}\|^2) \, ds \lesssim \left(\frac{|\Omega|^{\frac{2}{d}}}{\nu} + (t - t_0) \right) \frac{|\Omega|^{\frac{2}{d}}}{\nu} \|\mathbf{f}\|_{L^\infty(0, t; \mathbf{L}^2(\Omega))}^2,$$

which proves the $L^2_{\text{loc}}(0, \infty; L^2(\Omega))$ stability results. \square

REMARK 3.2. *The previous theorem proves the existence of an absorbing set for $[\mathbf{u}_h, \tilde{\mathbf{u}}]$ in $L^2(\Omega) \times L^2(\Omega)$ of radius Re . Let us stress the fact that any stabilized finite element formulation without a dynamic sub-grid model does not exhibit the sub-grid attractor and the pointwise (in time) sub-grid stability bounds.*

REMARK 3.3. *The algorithm (2.7) improves the crude Galerkin approximation by introducing sub-grid stability. However, it is not straightforward from the previous result how the sub-grid stability enhances the finite element stability, both in terms of pressure and velocities. We address this point below.*

In the next theorem we translate the sub-grid stability in terms of the finite element components, as it is usual for stabilized methods. The extra estimates for scheme (2.7) in the next theorem, that the Galerkin finite-element method does not provide, are weighted with a time-independent parameter $\tau_0 = \inf_{t \in (0, \infty)} \tau(t)$, i.e.

$$\tau_0^{-1} = \frac{C_s \nu}{h^2} + \frac{C_c \sup_{t \in (0, \infty)} \|\mathbf{u}_h(t)\|_{0, \ell}}{h |\Omega|^{\frac{1}{\ell}}}.$$

Observe that the parameter τ_0^{-1} is well-defined for a fixed $h > 0$ by using an inverse inequality $\|\mathbf{v}_h\|_{0, \ell} \lesssim h^{-(\frac{1}{2} - \frac{1}{\ell})} \|\mathbf{v}_h\|$ (for $2 \leq \ell \leq \infty$) and estimate (3.1). Thus, τ_0 does not degenerate to 0. Let us stress the fact that the introduction of the weighting parameter τ_0 comes from technical aspects in the subsequent analysis but the results apply to system (2.7) with the time-dependent expression of τ in (2.9).

THEOREM 3.3. *Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 . The algorithm (2.7) with $2 < \ell \leq \infty$ in (2.9) satisfies, for any $\bar{t} \geq t_0$,*

$$\tau_0^{\frac{1}{2}} \|\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)\|_{H_0^{-1}(t_0, \bar{t}; L^{q'}(\Omega))} \leq C,$$

for $q' = \frac{2\ell}{\ell-2}$. The case $\ell = 2$ satisfies

$$\tau_0^{\frac{1}{2}} \|\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)\|_{H_0^{-1}(t_0, \bar{t}; \mathbf{W}_0^{-1, (d+\varepsilon)'})} \leq C,$$

for a fixed $\varepsilon > 0$, where $(d + \varepsilon)'$ denotes its conjugate exponent, and C is a constant that depends on $(\mathbf{u}_0, \rho, \Omega)$. In particular, for $t_0 \rightarrow \infty$, C only depends on (ρ, Ω)

Proof. Recall that $\Pi_{V_h}(\cdot)$ is the orthogonal projection operator with respect the L^2 inner product. Let us pick $\mathbf{v}_h = \Pi_{V_h}(\mathbf{v})$ into the finite element equation, where the regularity of \mathbf{v} will be defined later on, integrate it over a finite interval $[t_0, \bar{t}]$ and multiply the resulting equation by the scalar value $\tau_0^{\frac{1}{2}}$. For simplification, let us also consider that $\mathbf{v}(\bar{t}) = \mathbf{v}(t_0) = 0$. We get:

$$\begin{aligned} (3.6) \quad & \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\Pi_{V_h}(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)), \mathbf{v}) \, ds \\ &= - \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \{(\partial_t \mathbf{u}_h, \mathbf{v}_h) + \nu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - b(\mathbf{u}_h, \mathbf{v}_h, \tilde{\mathbf{u}})\} \, ds. \end{aligned}$$

In the following, we bound the right-hand side terms in the finite element equation (3.6). The first term can be bounded using integration-by-parts in time and the definition of \mathbf{v} , in order to obtain:

$$\begin{aligned} (3.7) \quad & - \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\partial_t \mathbf{u}_h, \mathbf{v}_h) \, ds = \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\mathbf{u}_h, \partial_t \mathbf{v}_h) \, ds \lesssim \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \|\mathbf{u}_h\| \|\partial_t \mathbf{v}_h\| \, ds \\ & \lesssim \tau_0^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\mathbf{u}_h\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\partial_t \mathbf{v}_h\|^2 \, ds \right)^{\frac{1}{2}}. \end{aligned}$$

The convective term is bounded using Hölder's inequality for mixed norms (see [1]) as follows:

$$\begin{aligned}
(3.8) \quad - \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} b(\mathbf{u}_h, \mathbf{v}_h, \tilde{\mathbf{u}}) \, ds &\lesssim \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \|\tilde{\mathbf{u}}\| h^{-1} \|\mathbf{u}_h\|_{\ell} \|\mathbf{v}_h\|_q \, ds \\
&\lesssim \int_{t_0}^{\bar{t}} \tau^{-\frac{1}{2}} \|\tilde{\mathbf{u}}\| |\Omega|^{\frac{1}{\ell}} \|\mathbf{v}_h\|_q \, ds \\
&\lesssim |\Omega|^{\frac{1}{\ell}} \left(\int_{t_0}^{\bar{t}} \tau^{-1} \|\tilde{\mathbf{u}}\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\mathbf{v}_h\|_q^2 \, ds \right)^{\frac{1}{2}},
\end{aligned}$$

where we recall that $2 \leq \ell \leq \infty$ in the definition (2.9) of τ , whereas $q = \frac{2\ell}{\ell-2}$. Let us observe that $q \geq 2$. Finally, using an inverse inequality, we obtain

$$\begin{aligned}
(3.9) \quad - \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \nu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) \, ds &\lesssim \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} h^{-1} \nu \|\nabla \mathbf{u}_h\| \|\mathbf{v}_h\| \, ds \\
&\lesssim \left(\int_{t_0}^{\bar{t}} \nu \|\nabla \mathbf{u}_h\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\mathbf{v}_h\|^2 \, ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Combining (3.7)-(3.9), we get

$$\begin{aligned}
\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\Pi_{V_h}(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)), \mathbf{v}) \, ds &\leq \tau_0^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\mathbf{u}_h\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\partial_t \mathbf{v}_h\|^2 \, ds \right)^{\frac{1}{2}} \\
&\quad + |\Omega|^{\frac{1}{\ell}} \left(\int_{t_0}^{\bar{t}} \tau^{-1} \|\tilde{\mathbf{u}}\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\mathbf{v}_h\|_q^2 \, ds \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{t_0}^{\bar{t}} \nu \|\nabla \mathbf{u}_h\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\mathbf{v}_h\|^2 \, ds \right)^{\frac{1}{2}}.
\end{aligned}$$

In view of the above discussion, we consider $\mathbf{v} \in H_0^1(t_0, \bar{t}; \mathbf{L}^q(\Omega))$ to conclude that

$$(3.10) \quad \tau_0^{\frac{1}{2}} \|\Pi_{V_h}(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h))\|_{H_0^{-1}(t_0, \bar{t}; \mathbf{L}^{q'}(\Omega))} \leq C,$$

with q' being the conjugate of q and C involving the problem data $(\mathbf{u}_0, \rho, \Omega)$, by using the fact that $\Pi_{V_h}(\cdot)$ is a stable operator in L^s , with $1 \leq s \leq \infty$. In particular, for $t_0 \rightarrow \infty$, C only depends on (ρ, Ω) . Note that when $\ell = 2$, we have $q = \infty$, whose dual space is not identified with $L^1(\Omega)$. To bypass this problem, we use the Sobolev embedding $W_0^{1, d+\varepsilon}(\Omega) \hookrightarrow L^\infty(\Omega)$, where d is the space dimension, and $\varepsilon > 0$ a fixed number. Therefore, we have that

$$\tau_0^{\frac{1}{2}} \|\Pi_{V_h}(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h))\|_{H_0^{-1}(\bar{t}, t_0, \mathbf{W}_0^{-1, (d+\varepsilon)' }(\Omega))} \leq C,$$

when $\ell = 2$ and $(d + \varepsilon)'$ the conjugate of $(d + \varepsilon)$.

Our next step is to find a bound for the subscale part of $\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)$. For this we multiply the subscale equation by $\tau_0^{\frac{1}{2}}$, and integrate it over a finite interval $[t_0, \bar{t}]$. We get:

$$(3.11) \quad \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\Pi_{V_h}^\perp(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)), \mathbf{v}) \, ds = - \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\partial_t \tilde{\mathbf{u}} + \tau^{-1} \tilde{\mathbf{u}}, \mathbf{v}) \, ds.$$

For the right-hand side terms in the sub-grid equation (3.11), we proceed as follows:

$$\begin{aligned} - \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\partial_t \tilde{\mathbf{u}}, \mathbf{v}) \, ds &= \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\tilde{\mathbf{u}}, \partial_t \mathbf{v}) \, ds \lesssim \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \|\tilde{\mathbf{u}}\| \|\partial_t \mathbf{v}\| \, ds \\ &\lesssim \tau_0^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\tilde{\mathbf{u}}\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\partial_t \mathbf{v}\|^2 \, ds \right)^{\frac{1}{2}}, \\ - \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\tau^{-1} \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \, ds &\lesssim \left(\int_{t_0}^{\bar{t}} \tau^{-1} \|\tilde{\mathbf{u}}\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\tilde{\mathbf{v}}\|^2 \, ds \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\Pi_{V_h}^\perp (\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)), \mathbf{v}) \, ds &\lesssim \tau_0^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\tilde{\mathbf{u}}\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\partial_t \tilde{\mathbf{v}}\|^2 \, ds \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{t_0}^{\bar{t}} \tau^{-1} \|\tilde{\mathbf{u}}\|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\bar{t}} \|\tilde{\mathbf{v}}\|^2 \, ds \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, we have proved by selecting $\mathbf{v} \in H_0^1(t_0, \bar{t}; \mathbf{L}^2(\Omega))$ that

$$(3.12) \quad \tau_0^{\frac{1}{2}} \|\Pi_{V_h}^\perp (\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h))\|_{H_0^{-1}(t_0, \bar{t}; \mathbf{L}^2(\Omega))} \leq C.$$

Then, it is clear that from (3.10) and (3.12) that we have

$$\tau_0^{\frac{1}{2}} \|\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)\|_{H_0^{-1}(t_0, \bar{t}; \mathbf{L}^{q'}(\Omega))} \leq C,$$

where C only depends on (ρ, Ω) . Analogously, for $\ell = 2$, we arrive at

$$\|\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)\|_{H_0^{-1}(t_0, \bar{t}; \mathbf{W}_0^{-1, (d+\varepsilon)'})} \leq C,$$

for a fixed $\varepsilon > 0$. \square

REMARK 3.4. *The previous result proves the effectiveness of algorithm (2.7) as a stabilization technique. Both pressure and velocity stability that does not vanish with $\nu \rightarrow 0$ has been proved at all times. From the previous analysis, we can easily see that a quasi-static orthogonal sub-grid model, which consists in (2.7) but neglecting the sub-grid time derivative in (2.7d), also exhibits this kind of long-term stability. This is not true for quasi-static ASGS techniques, which do not even satisfy short term stability bounds for the semi-discrete problem in space (see [2]).*

REMARK 3.5. *Pressure stability in time is unclear even for Galerkin finite element approximations that are inf-sup stable, since there is no way to bound the time derivative of the fluid velocity in the appropriate space norms.*

REMARK 3.6. *The previous results bound a sum of pressure and convection terms, whereas it would be desirable a separate control of these two terms. This kind of result is not specific of our formulation, being a common feature of residual-based stabilization techniques. In fact, this is the case even for the steady Navier-Stokes equations (see e.g. [9]). Numerical evidence shows the effectiveness of residual-based stabilization techniques, even though separate bounds that would be effective for large Re have not been proved so far. A partial remedy could be the split version of the stabilization terms proposed in [11].*

Connected to the last remark, the next result provides an estimate for the convective term independent of the pressure term when $2 < \ell \leq \infty$. For the linearized problem and with divergence free advection velocities, this would allow us to obtain as well an estimate for the pressure gradient alone using Theorem 3.10. For the problem we consider, this could also be achieved introducing the pressure sub-grid scale and the additional control on the velocity divergence it provides, although we will not exploit this here (see [10, 11]). As far as we know, even though it is rather weak, this is the first time that a result of this kind is established.

COROLLARY 3.4. *There holds*

$$\tau^{\frac{1}{2}} \|\nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h)\|_{L^\infty(0,T; \mathbf{W}_0^{-1,s'}(\Omega))} \leq C$$

where s' is the conjugate of s such that $s = \frac{2\ell}{\ell-1}$ in two dimensions and $s = \frac{12\ell}{5\ell-6}$ in three dimensions, when $2 < \ell \leq \infty$.

Proof. Let us give the proof for the two-dimensional case only. For each $2 < \ell \leq \infty$, we can find $2 < r < \ell$ such that the interpolation inequality

$$\|\mathbf{u}_h\|_{0,r} \leq \|\mathbf{u}_h\|^{\frac{1}{2}}_{0,\ell} \|\mathbf{u}_h\|^{\frac{1}{2}}_{0,\ell}$$

holds, with $\frac{1}{2} + \frac{1}{\ell} = \frac{2}{r}$. Therefore, thanks to $\frac{1}{4} + \frac{1}{r} + \frac{1}{s} = 1$, we write, for all $\phi \in \mathbf{W}_0^{1,s}(\Omega)$,

$$\begin{aligned} \tau^{\frac{1}{2}} (\nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h), \phi) &\leq \tau^{\frac{1}{2}} \|\mathbf{u}_h\|_{0,r} \|\mathbf{u}_h\|_{0,4} \|\nabla \phi\|_{0,s} \\ &\leq \tau^{\frac{1}{2}} \|\mathbf{u}_h\|^{1/2}_{0,\ell} \|\mathbf{u}_h\|^{1/2}_{0,\ell} \|\mathbf{u}_h\|_{0,4} \|\nabla \phi\|_{0,s} \\ &\leq \left(\frac{h \|\mathbf{u}_h\|_{0,\ell}}{\nu + h \|\mathbf{u}_h\|_{0,\ell}(\Omega)} \right)^{1/2} h^{1/2} \|\mathbf{u}_h\|_{0,4} \|\phi\|_{1,s} \lesssim \|\phi\|_{1,s}, \end{aligned}$$

where in the last line we have used the inverse inequality $\|\mathbf{v}_h\|_{0,4} \lesssim h^{-\frac{1}{2}} \|\mathbf{v}_h\|_{0,2}$. \square

4. Absorbing set in $H^1(\Omega)$ and the global attractor for $d = 2$. In this section, we prove the existence of an absorbing set in $H^1(\Omega)$, which is the key result for the existence of a global attractor for algorithm (2.7). Let us introduce first the uniform Gronwall lemma (see e.g. [37]).

LEMMA 4.1 (Uniform Gronwall lemma). *Let x, μ, f be three positive locally integrable functions on (t_0, ∞) , such that $\partial_t x$ is locally integrable on (t_0, ∞) , and which satisfies*

$$\partial_t x \leq \mu x + f, \quad \text{in } t \geq t_0,$$

$$\int_t^{t+r} \mu(s) ds \leq a_1, \quad \int_t^{t+r} f(s) ds \leq a_2, \quad \int_t^{t+r} x(s) ds \leq a_3 \quad \text{in } t \geq t_0,$$

where r, a_1, a_2, a_3 are positive constant values. Then,

$$x(t+r) \leq \left(\frac{a_3}{r} + a_2 \right) \exp(a_1).$$

In order to get the bounds that lead to the existence of the $H^1(\Omega)$ absorbing set, let us introduce the scalar value

$$\tau_U^{-1} = \frac{C_s \nu}{h^2} + \frac{C_c U}{h},$$

where $U > 0$ is a bounded characteristic velocity of the problem. In particular, $U = \sup_{t \in (t_0, \infty)} |\Omega|^{-\frac{1}{\ell}} \|\mathbf{u}_h\|$ is a possible choice, since $\ell \geq 2$ and $\sup_{t \in (t_0, \infty)} \|\mathbf{u}_h\|$ has been bounded in Theorem 3.2. The long-term stability of the sub-grid velocity in the next theorem is weighted by $\tau_0^{\frac{1}{2}}$, whose introduction has been motivated by technical reasons. Again, the introduction of the weighting parameter τ_U is purely technical and the following results apply to system (2.7) with the time-dependent expression of τ in (2.9).

THEOREM 4.2 ($\mathbf{H}^1(\Omega)$ absorbing set). *Let $\Omega \subset \mathbb{R}^2$ have the elliptic regularity assumptions (2.13). Then, the solution $(\mathbf{u}_h, p_h, \tilde{\mathbf{u}})$ of problem (2.7), for $2 \leq \ell < \infty$, satisfies the long-term stability bound*

$$\limsup_{t \rightarrow \infty} (\nu \|\nabla \mathbf{u}_h\|^2 + \tau_U^{-1} \|\tilde{\mathbf{u}}\|^2) \lesssim \left(a_3 + \frac{a_2}{\bar{t}}\right) \exp(a_1),$$

with

(4.1a)

$$a_1 = \int_t^{t+\bar{t}} (\|\mathbf{f}\|^2 + U^4) \, ds \leq \bar{t} \left(\|\mathbf{f}\|_{L^\infty(0, \infty; \mathbf{L}^2(\Omega))}^2 + U^4 \right),$$

(4.1b)

$$a_2 = \int_t^{t+\bar{t}} (\nu \|\nabla \mathbf{u}_h\|^2 + \tau_U^{-1} \|\tilde{\mathbf{u}}\|^2) \, ds \lesssim \rho^2 \left(1 + \frac{\bar{t}\nu}{|\Omega|} \right),$$

(4.1c)

$$a_3 = \int_t^{t+\bar{t}} \nu^{-2} (\nu^{-2} \|\mathbf{u}_h\|^2 + 1) (\nu \|\nabla \mathbf{u}_h\|^2 + \tau_U^{-1} \|\tilde{\mathbf{u}}\|^2) \, ds \lesssim (\nu^{-4} \rho^2 + \nu^{-2}) a_2,$$

for any fixed $\bar{t} > 0$. This bound proves the existence of an absorbing set in $\mathbf{H}^1(\Omega)$ for the finite element fluid velocity and an absorbing set in $\mathbf{L}^2(\Omega)$ for $\tau_U^{-\frac{1}{2}} \tilde{\mathbf{u}}$.

Proof. Let us re-formulate system (2.7a), (2.7c), and (2.7d) in an appropriate way for the subsequent analysis, introducing the new variables \mathbf{z}_h and $\tilde{\mathbf{z}}$:

$$(4.2) \quad (\partial_t \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + (\mathbf{z}_h, \mathbf{v}_h) - b(\mathbf{u}_h, \mathbf{v}_h, \tilde{\mathbf{u}}) = (\mathbf{f}, \mathbf{v}_h),$$

$$(4.3) \quad \nu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{z}_h, \mathbf{v}_h),$$

$$(4.4) \quad (q_h, \nabla \cdot \mathbf{u}_h) - (\tilde{\mathbf{u}}, \nabla q_h) = 0,$$

$$(4.5) \quad (\partial_t \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + (\tilde{\mathbf{z}}, \tilde{\mathbf{v}}) + C_c \left(\frac{\|\mathbf{u}_h\|_\ell}{|\Omega|^{\frac{1}{\ell}} h} - \frac{U}{h} \right) (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = (\mathbf{f}, \tilde{\mathbf{v}}) - b(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{v}}),$$

$$(4.6) \quad \tau_U^{-1} (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + (\nabla p_h, \tilde{\mathbf{v}}) = (\tilde{\mathbf{z}}, \tilde{\mathbf{v}}).$$

First, we take $\mathbf{v}_h = \mathbf{z}_h$ in (4.2) and $\tilde{\mathbf{v}} = \tilde{\mathbf{z}}$ in (4.5), in order to get

$$(4.7a) \quad (\partial_t \mathbf{u}_h, \mathbf{z}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{z}_h) + \|\mathbf{z}_h\|^2 - b(\mathbf{u}_h, \mathbf{z}_h, \tilde{\mathbf{u}}) = (\mathbf{f}, \mathbf{z}_h),$$

$$(4.7b) \quad (\partial_t \tilde{\mathbf{u}}, \tilde{\mathbf{z}}) + \|\tilde{\mathbf{z}}\|^2 = (\mathbf{f}, \tilde{\mathbf{z}}) - b(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{z}}) - C_c \left(\frac{\|\mathbf{u}_h\|_p}{|\Omega|^{\frac{1}{p}} h} - \frac{U}{h} \right) (\tilde{\mathbf{u}}, \tilde{\mathbf{z}}).$$

Now, we can formally pick $\mathbf{v}_h = \partial_t \mathbf{u}_h$ in (4.3), and $\tilde{\mathbf{v}} = \partial_t \tilde{\mathbf{u}}$ in (4.6), i.e. pick smooth test functions as close as we want to these time derivatives, using density argument, and take limits. Doing that, we finally get:

$$(4.8a) \quad \frac{1}{2} \partial_t \nu \|\nabla \mathbf{u}_h\|^2 - (p_h, \nabla \cdot \partial_t \mathbf{u}_h) = (\mathbf{z}_h, \partial_t \mathbf{u}_h),$$

$$(4.8b) \quad \frac{1}{2} \partial_t \tau_U^{-1} \|\tilde{\mathbf{u}}\|^2 + (\nabla p_h, \partial_t \tilde{\mathbf{u}}) = (\tilde{\mathbf{z}}, \partial_t \tilde{\mathbf{u}}).$$

Note that we have used the fact that τ_U is constant in time; this technical reason prevents us from getting long-term subscale estimates multiplied by the time-dependent stabilization parameter τ that is used in the algorithm. Let us differentiate equation (4.4) with respect to t , and then take $q_h = p_h$:

$$(p_h, \nabla \cdot \partial_t \mathbf{u}_h) - (\partial_t \tilde{\mathbf{u}}, \nabla p_h) = 0.$$

We invoke this result in (4.7) and (4.8), obtaining:

$$(4.9) \quad \partial_t \left(\frac{\nu}{2} \|\nabla \mathbf{u}_h\|^2 + \frac{\tau_U^{-1}}{2} \|\tilde{\mathbf{u}}\|^2 \right) + \|\mathbf{z}_h\|^2 + \|\tilde{\mathbf{z}}\|^2 = (\mathbf{f}, \mathbf{z}_h) + (\mathbf{f}, \tilde{\mathbf{z}}) \\ - b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{z}_h) - b(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{z}}) + b(\mathbf{u}_h, \mathbf{z}_h, \tilde{\mathbf{u}}) - C \left(\frac{\|\mathbf{u}_h\|_p}{|\Omega|^{\frac{1}{p}} h} - \frac{U}{h} \right) (\tilde{\mathbf{u}}, \tilde{\mathbf{z}}).$$

Before controlling the right-hand side of (4.9) we introduce some technical tools. Let us define $\hat{\mathbf{u}} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ as the solution of the following Stokes problem

$$\begin{cases} -\nu \Delta \hat{\mathbf{u}} + \nabla \hat{p} &= \mathbf{g} := \mathbf{z}_h + \tilde{\mathbf{z}} + (\tau_\nu^{-1} - \tau_U^{-1}) \tilde{\mathbf{u}} & \text{in } \Omega, \\ \nabla \cdot \hat{\mathbf{u}} &= 0 & \text{in } \Omega, \\ \hat{\mathbf{u}} &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tau_\nu^{-1} := C_s h^{-2} \nu$ (see (2.14)). From (4.3), (4.4), (4.6), one can write

$$\begin{cases} \nu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) &= (\mathbf{z}_h, \mathbf{v}_h), \\ (q_h, \nabla \cdot \mathbf{u}_h) - (\tilde{\mathbf{u}}, \nabla q_h) &= 0, \\ \tau_\nu^{-1} (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + (\nabla p_h, \tilde{\mathbf{v}}) &= ((\tau_\nu^{-1} - \tau_U^{-1}) \tilde{\mathbf{u}} + \tilde{\mathbf{z}}, \tilde{\mathbf{v}}). \end{cases}$$

From Lemma 2.2, we know that $\nu^{\frac{1}{2}} \|\nabla(\hat{\mathbf{u}} - \mathbf{u}_h)\| + h^{-1} \nu^{\frac{1}{2}} \|\tilde{\mathbf{u}}\| \lesssim h \nu^{-\frac{1}{2}} \|\mathbf{g}\|$. Next, we want to bound $\|\Delta_h \mathbf{u}_h\|$ in terms of $\nu^{-1} \|\mathbf{g}\|$. Indeed, taking $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ solution of $\Delta \mathbf{u} = \Delta_h \mathbf{u}_h$, we obtain (see [25]):

$$\begin{aligned} \|\Delta_h \mathbf{u}_h\|^2 &\leq -(\nabla \mathbf{u}_h, \nabla \Delta_h \mathbf{u}_h) \\ &\leq -(\nabla(\hat{\mathbf{u}} - \mathbf{u}_h), \nabla \Delta_h \mathbf{u}_h) + (\Delta \hat{\mathbf{u}}, \Delta_h \mathbf{u}_h) \\ &\leq \|\Delta_h \mathbf{u}_h\| (h^{-1} \|\nabla(\hat{\mathbf{u}} - \mathbf{u}_h)\| + \|\Delta \hat{\mathbf{u}}\|) \\ &\leq C \|\Delta_h \mathbf{u}_h\| \nu^{-1} \|\mathbf{g}\|. \end{aligned}$$

This result, together with Lemma 2.2, allows to say that $\nu \|\Delta_h \mathbf{u}_h\| + h^{-2} \nu \|\tilde{\mathbf{u}}\| \lesssim \|\mathbf{g}\|$. On the other hand, using the expression of τ_U , we find:

$$\|\mathbf{g}\| \lesssim \|\mathbf{z}_h\| + \|\tilde{\mathbf{z}}\| + h^{-1} U \|\tilde{\mathbf{u}}\| \lesssim \|\mathbf{z}_h\| + \|\tilde{\mathbf{z}}\| + U^2 + \nu^{-1} \tau_U^{-1} \|\tilde{\mathbf{u}}\|^2.$$

Our goal now is to bound the right-hand side of (4.9). For every nonlinear term, we will repeatedly apply the results of Lemma 2.3 and Young's inequality. For the first nonlinear term

on the right hand side of (4.9), we use these ingredients and the fact that $\nu \|\Delta_h \mathbf{u}_h\| \lesssim \|\mathbf{g}\|$:

$$\begin{aligned} b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{z}_h) &\leq \|\mathbf{u}_h\|_{0,4} \|\nabla \mathbf{u}_h\|_{0,4} \|\mathbf{z}_h\| \lesssim \|\mathbf{u}_h\|^{\frac{1}{2}} \|\nabla \mathbf{u}_h\| \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}} \|\mathbf{z}_h\| \\ &\lesssim \frac{\nu^{-1}}{\delta} \|\mathbf{u}_h\| \|\nabla \mathbf{u}_h\|^2 \|\mathbf{g}\| + \delta \|\mathbf{z}_h\|^2 \\ &\lesssim \frac{\nu^{-2}}{\delta^3} \|\mathbf{u}_h\|^2 \|\nabla \mathbf{u}_h\|^4 + \delta \|\mathbf{g}\|^2 + \delta \|\mathbf{z}_h\|^2, \end{aligned}$$

and analogously

$$b(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{z}}) \lesssim \frac{\nu^{-2}}{\delta^3} \|\mathbf{u}_h\|^2 \|\nabla \mathbf{u}_h\|^4 + \delta \|\mathbf{g}\|^2 + \delta \|\tilde{\mathbf{z}}\|^2.$$

For the third nonlinear term, we use the standard inverse inequality, the expression for τ_ν and the result $\tau_\nu^{-1} \|\tilde{\mathbf{u}}\| \lesssim \|\mathbf{g}\|$, obtaining the following bound:

$$\begin{aligned} b(\mathbf{u}_h, \mathbf{z}_h, \tilde{\mathbf{u}}) &\leq \|\tilde{\mathbf{u}}\| \|\mathbf{u}_h\|_{0,4} \|\nabla \mathbf{z}_h\|_{0,4} + \|\tilde{\mathbf{u}}\| \|\nabla \cdot \mathbf{u}_h\|_{0,4} \|\mathbf{z}_h\|_{0,4} \\ &\lesssim h^{-1} \|\tilde{\mathbf{u}}\| \|\mathbf{u}_h\|_{0,4} \|\mathbf{z}_h\|_{0,4} \\ &\lesssim h^{-1} \|\tilde{\mathbf{u}}\| \|\mathbf{u}_h\|^{\frac{1}{2}} \|\nabla \mathbf{u}_h\|^{\frac{1}{2}} \|\mathbf{z}_h\|^{\frac{1}{2}} \|\nabla \mathbf{z}_h\|^{\frac{1}{2}} \\ &\lesssim \|\mathbf{u}_h\|^{\frac{1}{2}} h^{-\frac{1}{2}} \|\tilde{\mathbf{u}}\|^{\frac{1}{2}} \|\nabla \mathbf{u}_h\|^{\frac{1}{2}} h^{-1} \|\tilde{\mathbf{u}}\|^{\frac{1}{2}} \|\mathbf{z}_h\| \\ &\lesssim \frac{\nu^{-2}}{\delta^3} \|\mathbf{u}_h\|^2 \nu^{-1} \tau_\nu^{-1} \|\tilde{\mathbf{u}}\|^2 \|\nabla \mathbf{u}_h\|^2 + \delta \|\mathbf{g}\|^2 + \delta \|\mathbf{z}_h\|^2 \\ &\lesssim \frac{\nu^{-2}}{\delta^3} \|\mathbf{u}_h\|^2 (\nu^{-2} \tau_\nu^{-2} \|\tilde{\mathbf{u}}\|^4 + \|\nabla \mathbf{u}_h\|^4) + \delta \|\mathbf{g}\|^2 + \delta \|\mathbf{z}_h\|^2. \end{aligned}$$

Finally, by using the Sobolev embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$, i.e. $\|\mathbf{u}\|_p \leq C_P |\Omega|^{\frac{1}{p}} \|\nabla \mathbf{u}\|$ for $d = 2$, and the expression of τ_ν , we bound the last term on the right hand side of (4.9) as follows:

$$\left(\frac{\|\mathbf{u}_h\|_p}{|\Omega|^{\frac{1}{p}} h} - \frac{U}{h} \right) (\tilde{\mathbf{u}}, \tilde{\mathbf{z}}) \leq \frac{1}{\delta} \|\nabla \mathbf{u}_h\|^4 + \frac{1}{\delta} U^4 + \frac{1}{\delta} \nu^{-2} \tau_\nu^{-2} \|\tilde{\mathbf{u}}\|^4 + \delta \|\tilde{\mathbf{z}}\|^2.$$

For the force terms, we simply have

$$(\mathbf{f}, \mathbf{z}_h) + (\mathbf{f}, \tilde{\mathbf{z}}) \lesssim \frac{1}{\delta} \|\mathbf{f}\|^2 + \delta \|\mathbf{z}_h\|^2 + \delta \|\tilde{\mathbf{z}}\|^2.$$

The above bounds applied to (4.9), picking δ small enough, yield

$$\begin{aligned} &\partial_t (\nu \|\nabla \mathbf{u}_h\|^2 + \tau_U^{-1} \|\tilde{\mathbf{u}}\|^2) + \|\mathbf{z}_h\|^2 + \|\tilde{\mathbf{z}}\|^2 \\ &\lesssim \nu^{-2} (\nu^{-2} \|\mathbf{u}_h\|^2 + 1) (\tau_\nu^{-2} \|\tilde{\mathbf{u}}\|^4 + \nu^2 \|\nabla \mathbf{u}_h\|^4) + U^4 + \|\mathbf{f}\|^2 \\ &\lesssim \nu^{-2} (\nu^{-2} \|\mathbf{u}_h\|^2 + 1) (\tau_\nu^{-1} \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2) (\tau_U^{-1} \|\tilde{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}_h\|^2) + U^4 + \|\mathbf{f}\|^2. \end{aligned}$$

We finish the proof using the uniform Gronwall lemma over the previous inequality, with constants (4.1). \square

REMARK 4.1. *The previous stability bounds lead to an absorbing set in $\mathbf{H}^1(\Omega)$ for the finite element component of the velocity. With regard to the sub-grid scale, this theorem proves that $\tau_U^{-\frac{1}{2}} \tilde{\mathbf{u}}$ also exhibits an absorbing set in $\mathbf{L}^2(\Omega)$, which can only be obtained for dynamic sub-grid models.*

REMARK 4.2. *With regard to the norms involved, the previous results are stronger than those in Theorem 3.2. However, the radius of the absorbing set in Theorem 3.2 is much smaller than the one for Theorem 4.2 for large Re. Thus, from a numerical point of view, in which constants do matter, the $\mathbf{L}^2(\Omega)$ results are stronger.*

REMARK 4.3. *The previous result allows to say that there is a ball in $\mathbf{H}^1(\Omega)$ that absorbs all orbits for large enough time values. Due to the Rellich-Kondrachov embedding theorem, i.e. $H^1(\Omega) \hookrightarrow L^2(\Omega)$, the two-dimensional Navier-Stokes equations posses a compact absorbing set in $\mathbf{L}^2(\Omega)$. Thus, the operators $\mathcal{S}(t)$ are uniformly compact for t large enough, and the existence of a compact global attractor can be proved (see [37, Theorem 1.1])*

5. Conclusions. We have presented a finite element approximation of the Navier-Stokes equations with numerical sub-grid scale modeling for which the results obtained here are easily summarized: we have been able to prove that the long term behavior is similar to what is found for the pure Galerkin method, plus additional control on the velocity sub-grid scales. In particular, we have shown that \mathbf{u}_h is bounded in $L^2(\Omega)$ for all time *and so is the velocity sub-grid scale $\tilde{\mathbf{u}}$* , that in 2D the spatial dissipation associated to \mathbf{u}_h is bounded in $L^2(0, \infty)$ *and so is the dissipation associated to $\tilde{\mathbf{u}}$* , and that \mathbf{u}_h has an absorbing set in $L^2(\Omega)$ *and so does $\tilde{\mathbf{u}}$* . For \mathbf{u}_h and in the 2D case, the absorbing set can be shown to be a global attractor using classical arguments.

The *benefit* of our approach is that additional control on the pressure and the convective term can be recovered from the stability obtained for the velocity sub-grid scales. The key point, and in some sense the essence of stabilized finite element methods for convection dominated flows, is that this control remains meaningful for $\nu \rightarrow 0$.

This last issue brings us to discuss the *limitations* of our analysis. As for all stabilized formulations we are aware of, *full* control on the pressure is not obtained (not even for the stationary Oseen problem), but only the sum of the pressure gradient and the convective term can be shown to be stable. In practice, however, this seems to be enough, although, as far as we know, no theoretical explanation has been provided. We have however provided a weak estimate in this direction, showing that some control can be proved for the convective term and the pressure gradient alone. Another limitation of our analysis is that we have needed to assume that the advection velocity is \mathbf{u}_h , and not $\mathbf{u}_h + \tilde{\mathbf{u}}$, and that we have had to take a constant stabilization parameter, whereas in practice it is computed from local values (at least at the element level).

Let us stress also that *the key* for being able to prove our stability estimates is twofold: the velocity sub-grid scale $\tilde{\mathbf{u}}$ *needs to be time dependent and orthogonal to the finite element space*. These ideas were introduced in [10] and we have used them in an essential way in the analysis presented herein.

The next issue we wish to consider is the design of time integration schemes that preserve the stability results proved here for the time-continuous case, particularly considering that the time integration of \mathbf{u}_h and of $\tilde{\mathbf{u}}$ will probably have different requirements. This is, however, the subject of future research.

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